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Associated relaxation time and intensity correlation function of a bistable system driven by cross-correlation additive and multiplicative coloured noise sources

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Abstract The associated relaxation time and the intensity correlation function of a bistable system driven by an additive and a multiplicative coloured noise with coloured cross-correlation are investigated. Using the Novikov theorem and the projection operator method, the analytic expressions of the stationary probability distribution $P_{st}(x)$, the relaxation time T_c, and the normalized correlation function $C(s)$ of the system are obtained. The effects of the noise intensity, the cross-correlation strength λ and the cross-correlation time τ are discussed. By numerical computation, it is found that the cross-correlation strength $|\lambda|$ and the quantum noise intensity D decrease the relaxation of the system from unstable points. The cross-correlation time τ delays relaxation of the system from unstable points. The cross-correlation strength λ and the cross-correlation time τ can alter the effects of the pump noise intensity Q. Thus, the relaxation time T_c is a stochastic resonant phenomenon, and distribution curves exhibit a single-maximum structure.

PACS. 05.40.-a Fluctuation phenomena – 02.50.-r Probability theory, stochastic processes – 05.10.Gg Stochastic analysis methods

1 Introduction

A bistable system with noise is a typical and important problem in statistical mechanics. It is related to many practical problems, including quenching phenomena [1–3], bistable optical systems [4, 5], stochastic resonant phonomenona [6–9], etc. In most previous work, noise sources are usually treated as uncorrelated random variables, since it is usually assumed that they have different origins. However, in some practical cases, sources of noise may have a common origin, and hence can be correlated [11, 12]. There are other situations where strong external noise can engender changes in the internal structure of a system so that the internal noise and the external noise should be independent [12–15]. Bistable systems with correlation noise terms are the subject of other studies [15–27]. Hanggi et al. first investigated colour effects in the activation rate of a bistable system [14]. Marchi et al. studied the resonant activation for a bistable system driven by an additive and a multiplicative noise [17]. Using the Novikov theorem and the steady-state value of the deterministic theory, Jia and Li analyzed the steady-state properties of the bistable kinetic model with cross-correlation additive and multiplicative white noise [28].

The associated relaxation time and the intensity correlation function are important physical quantities to characterize the statistical behaviour of a stochastic process, and hence are usually used to describe the fluctuation behaviour of a nonlinear system [29]. Research into the problem has shown that the associated relaxation time and the intensity correlation function for nonlinear stochastic systems are important physical features [30–33]. Applying the projection operator method, Xie and Mei investigated dynamical properties of a bistable kinetic model with correlated noise [34]. Mei et al. considered the effects of cross-correlated white noise sources on the relaxation time and the correlation function of a bistable system [35, 36]. They described the statistical properties of a bistable system with cross-correlation white noise sources. Considering two input signals that consist of an additive and a purely multiplicative random signal, Borromeo and Marchesoni investigated asymmetric probability densities in symmetrically modulated bistable devices [37, 38]. In their work it is found that the correlation between an additive and a multiplicative noise plays an important role in the processes of a nonlinear stochastic system. Recently, as the subject matures, attention has turned to stochastic systems with cross-correlation additive and multiplicative coloured noise sources. Ling et al. investigated the

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moments of the intensity of a single-mode laser driven by additive and multiplicative coloured noise sources with coloured cross-correlation [39]. Jin et al. [40] consider the relaxation time of a single-mode dye laser system driven by cross-correlation additive and multiplicative white noise sources.

The purpose of this work is to consider the effects of the cross-correlated coloured noise sources on the associated relaxation time, and on the intensity correlation function of a bistable system. In Section 2, the approximate Fokker-Planck equation (AFPE) is introduced for a bistable system with cross-correlation additive and multiplicative coloured noise sources. This is solved the AFPE for the stationary probability distribution (SPD). By using the projection operator method — in which the effects of the memory kernels are taken into account — the analytic expressions of the associated relaxation time and the normalized correlation function of a bistable system with cross-correlation coloured noise sources are derived. In Section 3, based on the numerical results, the relaxation time and the correlation function, the effects of the cross-correlation strength and the correlation time, and the stochastic resonant activation for the bistable system, are discussed. Thus, the important effects of cross-correlation coloured noise sources to the statistical properties of a bistable system are demonstrated.

2 Stationary probability distribution, relaxation time, and correlation function

Consider a conventional, symmetric, bistable, kinetic system driven by cross-correlation additive and multiplicative coloured noise, in which characteristics of the cross-correlation time and the self-correlation time of the noise sources may be different. The Langevin equation of this general system is

$$\frac{dx}{dt} = x - x^3 + x\xi(t) + \eta(t). \tag{1}$$

Here $\xi(t)$ and $\eta(t)$ are zero-mean Gaussian noise sources, whose statistical properties are

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \tag{2}$$

$$\langle \xi(t)\xi(t') \rangle = \frac{D}{\tau_1} \exp\left(-\frac{|t-t'|}{\tau_1}\right), \tag{3}$$

$$\langle \eta(t)\eta(t') \rangle = \frac{Q}{\tau_2} \exp\left(-\frac{|t-t'|}{\tau_2}\right), \quad \text{and} \tag{4}$$

$$\langle \xi(t)\eta(t') \rangle = \langle \eta(t)\xi(t') \rangle = \frac{\lambda\sqrt{DQ}}{\tau_3} \exp\left(-\frac{|t-t'|}{\tau_3}\right), \tag{5}$$

where D and Q are the multiplicative coloured noise and the additive coloured noise intensity, respectively. τ_1 and τ_2 are the self-correlation time of the multiplicative noise and the additive noise, respectively. τ_3 is the cross-correlation time of the multiplicative and additive coloured noise sources.

By virtue of the Novikov theorem [41], Fox's approach [42], and the ansatz of Hanggi et al. [43], the approximate Fokker-Planck equation corresponding to equation (1) with equations (2–5) is obtained [16,27,28]:

$$\frac{\partial P(x,t)}{\partial t} = L_{FP} P(x,t), \tag{6}$$

$$L_{FP} = -\frac{\partial}{\partial x} f(x) + \frac{\partial^2}{\partial x^2} G(x), \tag{7}$$

where

$$f(x) = x - x^3 + \frac{Dx}{1+2\tau_1} + \frac{\lambda\sqrt{DQ}}{1+2\tau_3}, \tag{8}$$

and

$$G(x) = \frac{Dx^2}{1+2\tau_1} + \frac{2\lambda\sqrt{DQ}x}{1+2\tau_3} + \frac{Q}{1+2\tau_2}. \tag{9}$$

Note that since $\tau_1 \geq 0$, $\tau_2 \geq 0$, and $\tau_3 \geq 0$ satisfy the approximate Fokker-Planck equation (6) when $1+2\tau_1 > 0$, $1+2\tau_2 > 0$, and $1+2\tau_3 > 0$, there is no restriction on τ_1, τ_2, and τ_3 [25]. Now consider the case of the self-correlation time and the cross-correlation time satisfying $\tau_1 = \tau_2 = \tau_3 = \tau$, then

$$f(x) = x - x^3 + \frac{Dx}{1+2\tau} + \frac{\lambda\sqrt{DQ}}{1+2\tau}, \tag{10}$$

and

$$G(x) = \frac{Dx^2}{1+2\tau} + \frac{2\lambda\sqrt{DQ}x}{1+2\tau} + \frac{Q}{1+2\tau}. \tag{11}$$

The stationary probability distribution of the system can be obtained from equation (6) with equations (10, 11):

$$\begin{aligned} P_{st}(x) = N &\left(\frac{Dx^2}{1+2\tau} + \frac{2\lambda\sqrt{DQ}x}{1+2\tau} + \frac{Q}{1+2\tau}\right)^{\beta_1} \\ &\times \exp\left[-\frac{1+2\tau}{2D}x^2 + 2\lambda(1+2\tau)\sqrt{\frac{Q}{D^3}}x\right] \\ &\times \exp\left[\beta_2 \arctan\left(\frac{\sqrt{\frac{D}{Q}}x + \lambda}{\sqrt{1-\lambda^2}}\right)\right] \end{aligned} \tag{12}$$

for $0 \leq |\lambda| < 1$; and

$$\begin{aligned} P_{st}(x) = N &\left(\frac{Dx^2}{1+2\tau} + \frac{2\lambda\sqrt{DQ}x}{1+2\tau} + \frac{Q}{1+2\tau}\right)^{\beta_1} \\ &\times \exp\left[-\frac{1+2\tau}{2D}x^2 + 2\lambda(1+2\tau)\sqrt{\frac{Q}{D^3}}x\right] \\ &\times \exp\left[\frac{-(1+2\tau)}{Dx + \lambda\sqrt{DQ}}\right] \end{aligned} \tag{13}$$

for $|\lambda| = 1$, where

$$\beta_1 = \frac{1+2\tau}{2D}\left[1 + \frac{Q}{D}\left(1-4\lambda^2\right)\right] - \frac{1}{2}, \tag{14}$$

$$\beta_2 = -\frac{\lambda(1+2\tau)}{D\sqrt{1-\lambda^2}}\left[1 + \frac{3Q}{D} - \frac{4Q\lambda^2}{D}\right], \tag{15}$$

and N is the normalization constants for equations (12) and (13). The normalization constant N is determined by

$$\int_{-\infty}^{\infty} P_{st}(x)dx = 1. \quad (16)$$

The expectation values of the nth power of the state variable x are given by

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P_{st}(x)dx. \quad (17)$$

For a general stochastic process for which a steady state exists, the stationary correlation function is defined by

$$C(s) = \langle \delta x(t+s)\delta x(t)\rangle_{st} = \lim_{t\to\infty} \langle \delta x(t+s)\delta x(t)\rangle, \quad (18)$$

where $\delta x(t) = x(t) - \langle x(t)\rangle$. The normalized correlation function is

$$C(s) = \frac{\langle \delta x(t+s)\delta x(t)\rangle_{st}}{\langle (\delta x)^2\rangle_{st}}. \quad (19)$$

The associated relaxation time which describes the fluctuation decay of the dynamical variable x is defined by

$$T_c = \int_0^{\infty} C(t)dt. \quad (20)$$

By using the projection operator method [35], the zero-order approximation for the relaxation time is given by

$$T_c = \gamma_0^{-1} = \frac{\langle (\delta x)^2\rangle_{st}}{\langle G(x)\rangle_{st}}. \quad (21)$$

Similarly, the first-order approximation for the relaxation time is given by

$$T_c = \left[\gamma_0 + \frac{\eta_1}{\gamma_1}\right]^{-1}, \quad (22)$$

where

$$\eta_1 = \frac{\langle G(x)f'(x)\rangle_{st}}{\langle (\delta x)^2\rangle_{st}} + \gamma_0^2, \quad (23)$$

and

$$\gamma_1 = -\frac{\langle G(x)[f'(x)]^2\rangle_{st}}{\eta_1\langle (\delta x)^2\rangle_{st}} + \frac{\gamma_0^3}{\eta_1} - 2\gamma_0. \quad (24)$$

Employing equations (10–17, 21, 23) and (24), gives

$$\gamma_0 = \frac{b_1 k_2}{\langle (x)^2\rangle_{st} - \langle x\rangle_{st}^2}, \quad (25)$$

$$\eta_1 = \frac{b_1\left[(1+Db_1)k_2 - 3k_4\right]}{\left[\langle (x)^2\rangle_{st} - \langle x\rangle_{st}^2\right]} + \gamma_0^2, \quad (26)$$

and

$$\gamma_1 = \frac{-b_1}{\eta_1\left[\langle (x)^2\rangle_{st} - \langle x\rangle_{st}^2\right]} \times \left[(1+Db_1)^2 k_2 - 6(1+Db_1)k_4 + 9k_6\right] + \frac{\gamma_0^3}{\eta_1} - 2\gamma_0; \quad (27)$$

where

$$b_1 = \frac{1}{1+2\tau}, \quad (28)$$

$$k_2 = D\langle x^2\rangle_{st} + 2\lambda\sqrt{DQ}\langle x\rangle_{st} + Q, \quad (29)$$

$$k_4 = D\langle x^4\rangle_{st} + 2\lambda\sqrt{DQ}\langle x^3\rangle_{st} + Q\langle x^2\rangle_{st}, \quad \text{and} \quad (30)$$

$$k_6 = D\langle x^6\rangle_{st} + 2\lambda\sqrt{DQ}\langle x^5\rangle_{st} + Q\langle x^4\rangle_{st}. \quad (31)$$

Here, we see that the zero-order approximation of the relaxation time $T_c = \gamma_0^{-1}$ is in good agreement with the result calculated via the Stratonovich-like ansatz in reference [44]. When $\lambda = 0$ and $Q = 0$, the above results reduce to equations (2.29–2.31), as presented in reference [30]. In other words, the Stratonovich-like ansatz completely neglects the memory kernel.

Applying the projection operator method and performing the converse Laplace transformation [35], the stationary correlation function is

$$C(s) = \beta\exp(-\pi_- s) + (1-\beta)\exp(-\pi_+ s), \quad (32)$$

where

$$\beta = \frac{\gamma_1 - \pi_-}{\pi_+ - \pi_-}, \quad (33)$$

and

$$\pi_\pm = \frac{\gamma_0 + \gamma_1}{2} \pm \frac{1}{2}\sqrt{(\gamma_0 - \gamma_1)^2 - 4\eta_1}. \quad (34)$$

3 Discussion and conclusions

Equation (1) represents a physical model of a bistable system driven by cross-correlation additive and multiplication coloured noise sources corresponding to the potential $V(x) = -x^2/2 + x^4/4$. The statistical properties of this stochastic system are obtained by numerical solution of the equation.

The relaxation time gives dynamical information about the time scale of the evolution of a spontaneous fluctuation in the steady state, which reflects the evolution velocity of the system from an arbitrary initial state to the steady state [29]. The relaxation time distribution curves of the bistable system are plotted in Figures 1–3

In Figure 1a, the $T_c - \lambda$ curves are symmetrical about the axis $\lambda = 0$. When $|\lambda|$ is smaller, the values of the correlation function increase as τ increases, while for a fixed τ the size of the correlation function is almost unchanged when $|\lambda|$ increases. However, when $|\lambda|$ is larger, the values of the correlation function decrease rapidly as $|\lambda|$ increases. Figure 1b shows that the effects of the cross-correlation strength λ are only determined by the absolute values of λ. It is seen that when $|\lambda|$ is smaller, the relaxation time increases as τ increases; when $|\lambda|$ is larger, the relaxation time decreases as τ increases.

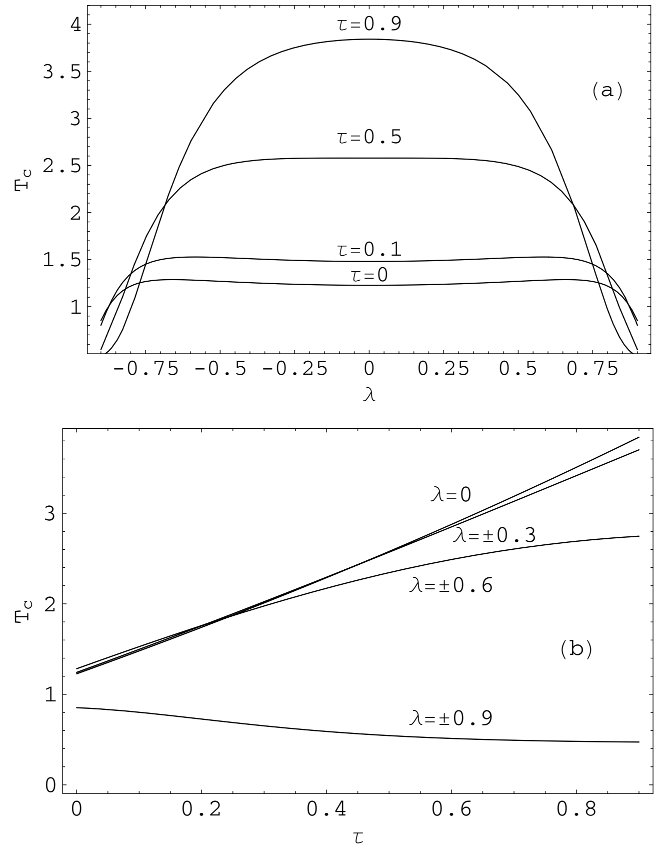

Figure 1. (a) The relaxation time T_c as a function of the cross-correlation strength λ for $D = 1$ and $Q = 0.25$. τ is 0, 0.1, 0.5 and 0.9 respectively. (b) The relaxation time T_c as a function of the correlation time τ for $D = 1$ and $Q = 0.25$. λ is 0, ± 0.3, ± 0.6 and ± 0.9, respectively.

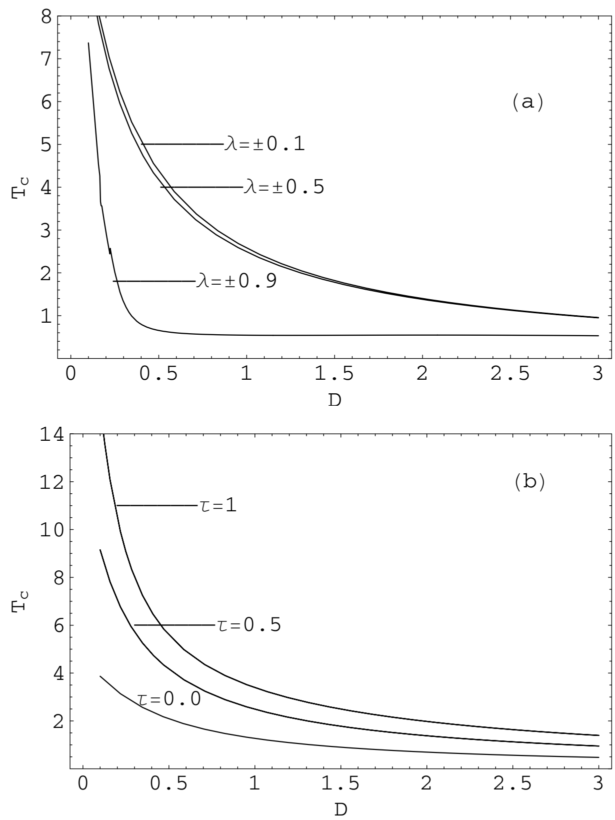

Figure 2. The relaxation time T_c as a function of the noise intensity D for $Q = 0.25$. (a) $\tau = 0.5$, λ is ± 0.1, ± 0.5, and ± 0.9, respectively. (b) $\lambda = \pm 0.5$, τ is 0, 0.5, and 1, respectively.

Figures 2 and 3 show the effects of the pump noise intensity D and the quantum noise intensity Q. An interesting stochastic resonant phenomenon for a bistable system is also seen. In Figure 2, the relaxation time T_c decreases monotonously as the noise intensity D increases. D expedites the relaxation of the system from unstable points, which when $D < Q$, the effects are most obvious; when $D > Q$, the effects are dampened. In Figure 2a, the correlation time τ is fixed to be 0.5: the relaxation time T_c decreases as the cross-correlation strength $|\lambda|$ increases. In Figure 2b, $|\lambda|$ is fixed to be 0.5: the relaxation time T_c increases as τ increases. Figure 3 shows that when both $|\lambda|$ and τ take smaller values, T_c decreases monotonously as the noise intensity Q increases. When both $|\lambda|$ and τ take larger values, the T_c distribution curves exhibit a single-maximum structure, and a stochastic resonant phenomenon occurs. When τ is fixed (Fig. 3a) the height of the peak of T_c decreases as $|\lambda|$ increases. In contrast, when $|\lambda|$ is fixed (Fig 3b) the height of the peak of T_c increases as τ increases.

The correlation function $C(s)$ describes the dynamical fluctuation decay of the variable x with time in the steady state, which reflects the related activity between two states at different time. Figure 4 shows that $C(s)$ decreases exponentially as the decay time s increases. In Figure 4a, τ is fixed at $\tau = 0.2$; the size of the correlation function decreases as $|\lambda|$ increases, and the effects of λ are only determined by the magnitude of λ. In Figure 4b, $|\lambda|$ is fixed at $\lambda = 0.2$; $C(s)$ increases as τ increases. In Figure 4c, τ is fixed at $\lambda = 0.8$; $C(s)$ decreases as τ increases.

From the above, we conclude that cross-correlation additive and multiplicative coloured noise sources play important roles in a bistable system. The cross-correlation strength $|\lambda|$ weakens the related activity between two states at different times, accelerates the evolution velocity, shortens the evolution time of the system from an arbitrary initial state to the stable state, and enhances the stability of the bistable system in the steady state. In contrast, the correlation time τ enhances the related activity between two states at different times, decrease the evolution velocity, delays the evolution time of the system from an arbitrary initial state to the stable state, and reduces the stability of the bistable system in the steady state. The noise intensity D and the cross-correlation strength $|\lambda|$ expedites the relaxation of the system from unstable points, but the correlation time τ delays the relaxation of the system from unstable points, which when $D < Q$, the effects are most obvious; when $D > Q$, the effects are dampened. The cross-correlation strength $|\lambda|$ and the

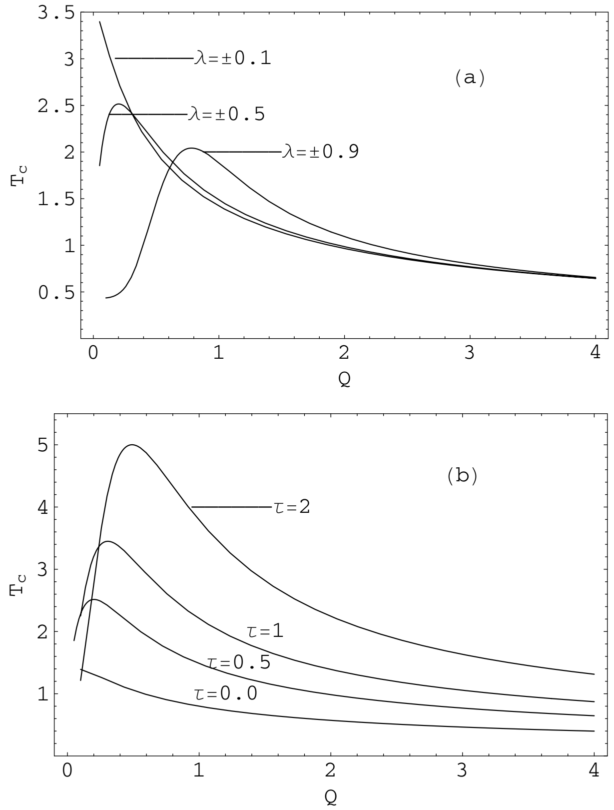

Figure 3. The relaxation time T_c as a function of the noise intensity Q for $D = 1$. (a) $\tau = 0.5$, λ is ± 0.1, ± 0.5, and ± 0.9, respectively. (b) $\lambda = \pm 0.5$, τ is 0, 0.5, 1, and 2, respectively.

correlation time τ can alter the effects of the noise intensity Q. Thus, the relaxation time T_c is a stochastic resonant phenomenon, and $T_c - Q$ curves exhibit a single-maximum structure, where the noise intensity $Q = Q_0$ at resonance. When $|\lambda|$ and τ increase, Q_0 becomes larger. The correlation strength $|\lambda|$ enhances the resonance of the relaxation time; by contrast, the correlation time τ dampens the resonance the relaxation time. In some interrelated practical problems, such as quenching phenomena, optical bistable systems, stochastic resonances, new applications to nanodevices [46], etc., the stability of the bistable system in the steady state can be improved by choosing suitable values for $|\lambda|$, τ, D, and Q.

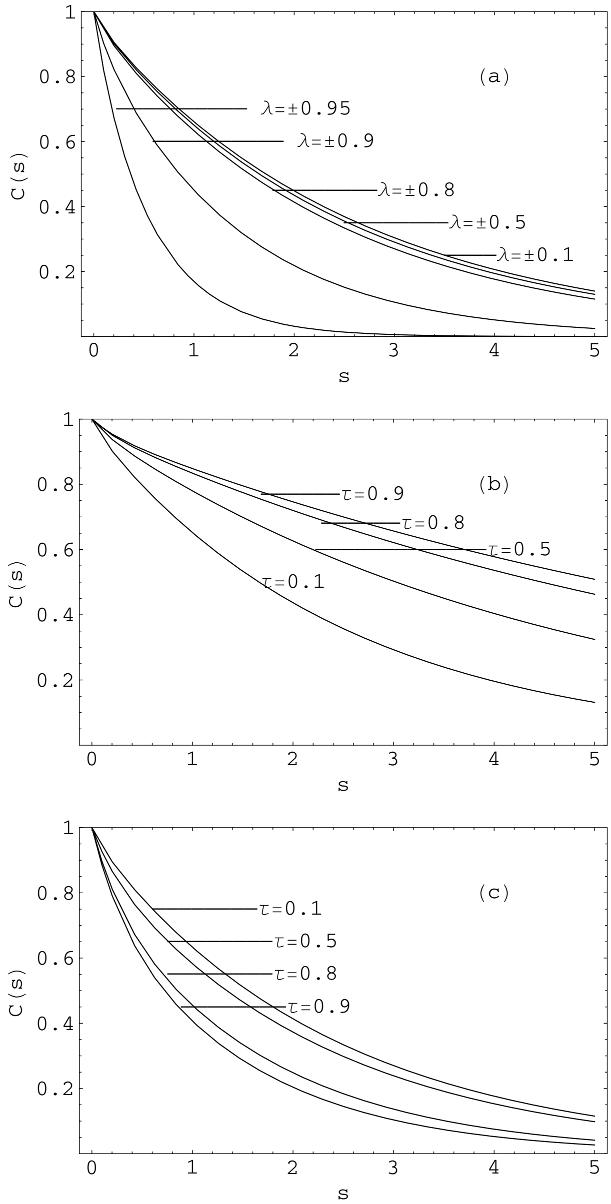

Figure 4. The normalized correlation function $C(s)$ as a function of decay time s for $D = 1$ and $Q = 0.25$. (a) $\tau = 0.2$. λ is ± 0.1, ± 0.5, ± 0.8, ± 0.9, and ± 0.95, respectively. (b) $\lambda = 0.2$. τ is 0.1, 0.5, 0.8, and 0.9, respectively. (c) $\lambda = 0.8$. τ is 0.1, 0.5, 0.8, and 0.9, respectively.

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